Exercise 1. Show the 3-transitivity of the Möbius group: if $(p_1, p_2, p_3) \in \partial\mathbb{D}$ and $(q_1, q_2, q_3) \in \partial\mathbb{D}$ are two couple of three distinct points in increasing order (that is, $(p_1, p_2, p_3) = (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ with $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$), then there exists a unique map $f \in \mathscr{M}_+(\mathbb{D})$ such that $f(p_i) = p_i$ for all $1 \leq i \leq 3$.

Exercise 2 (Another approach to the problem of Plateau)**.** The goal of the exercise is to show that the area functional $A : W^{1,2}(\mathbb{D}, \mathbb{R}^3)$ defined by

$$A(u) = \int_{\mathbb{D}} |\partial_x u \times \partial_y u| dx\, dy$$

is lower semi-continuous for the weak convergence in $W^{1,2}$.

1. Show that for all $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^3)$, we have

$$A(u) = \sup\left\{\int_{\mathbb{D}} \langle \partial_x u \times \partial_y u, \varphi\rangle dx\, dy : \ \varphi \in \mathscr{D}(\mathbb{D}, \mathbb{R}^3),\ \ \|\varphi\|_{\mathrm{L}^\infty(\mathbb{D})} \leq 1\right\}.$$

2. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,2}(\mathbb{D}, \mathbb{R}^3)$ weakly converges to $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^3)$ Show that for all $\varphi \in \mathscr{D}(\mathbb{D}, \mathbb{R}^3)$ such that $\|\varphi\|_{\mathrm{L}^\infty(\mathbb{D})} \leq 1$, we have

$$\lim_{n\to\infty} \int_{\mathbb{D}} \langle \partial_x u_n \times \partial_y u_n, \varphi\rangle dx\, dy = \int_{\mathbb{D}} \langle \partial_x u \times \partial_y u, \varphi\rangle dx\, dy.$$

3. Conclude the proof of the lower semi-continuity.

4. Consider for all $\varepsilon > 0$ the functional

$$A_\varepsilon(u) = A(u) + \varepsilon\, E(u) = A(u) + \frac{\varepsilon}{2}\int_{\mathbb{D}} |\nabla u|^2 dx\, dy.$$

   Show that for all $\varepsilon > 0$, the function A_ε is lower semi-continuous for the weak convergence in $W^{1,2}$.

Remark 1. This approach is the basis of an alternative proof of the problem of Plateau where we minimise A_ε and then we make $\varepsilon \to 0$. This method is known as the viscosity method and it has been applied to a wide variety of variational problems where compactness is lacking.

Exercise 3 (The Douglas functional)**.** For all $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$, define

$$D(u) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|u(e^{i\theta}) - u(e^{i\varphi})|^2}{4\sin^2\left(\frac{\theta-\varphi}{2}\right)} d\theta\, d\varphi.$$

Show that D is equivalent to the homogenous $H^{1/2}$ norm on S^1 given by

$$\|u\|_{\mathrm{H}^{1/2}(S^1)} = \left(\sum_{n\in\mathbb{Z}} |n||\widehat{u}(n)|^2\right)^{\frac{1}{2}},$$

where

$$\widehat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)e^{-in\theta} d\theta$$

is the Fourier coefficient.

Remark 2. This is the functional that Douglas originally used to solve the problem of Plateau. The Douglas functional actually coincides with the Dirichlet energy E, but the proof is involved and we will only see it in the next series.